Hence the migration velocity

$$v = \frac{l}{2m\sqrt{(3kT)}} \cdot \frac{2M+m}{\sqrt{(M)}} \quad \text{and} \quad 1/v\,\frac{dv}{dm} = \frac{1}{2M} \cdot \frac{2M-m}{2M+m}.$$

For lithium, where M_1 and M_2 are 6 and 7 if $m = 18$, there should be a difference of 1·24 per cent. in the migration velocity. For chlorine with $M_1 = 35$ and $M_2 = 37$, it should be nearly 1·7 per cent. If a solution were electrolysed in some medium like gelatine, the head of the column should contain one isotope only, and it is hoped shortly to carry out experiments to test this. In the above calculation, of course, the effect of hydration and association have not been taken into account.

One other point to which it may be worth while to refer is the very small energy difference between atoms if their atomic weights approach whole numbers closely. Quite apart from positive rays, chemical evidence shows that the difference between the mass of one carbon atom and three helium atoms is so small that the energy difference cannot exceed $4{\cdot}5 \,.\, 10^{-7}$ ergs per atom, *i.e.*, about 1/30 of the energy of the average radio-active change. If this is true, it somewhat curtails the store of intra-atomic energy, which it is hoped will one day become available.

A Generalisation of Weyl's Theory of the Electromagnetic and Gravitational Fields.

By A. S. Eddington, F.R.S.

(Received February 19, 1921.)

1. In the non-Euclidean geometry of Riemann, the metric is defined by certain quantities, $g_{\mu\nu}$, which are identified by Einstein with the potentials of the gravitational field. H. Weyl has shown that, on removing a rather artificial restriction in Riemann's geometry, the expression for the metric includes also terms which are identified with the four potentials of the electromagnetic field. I believe that Weyl's geometry, far-reaching though it is, yet suffers from an unnecessary and harmful restriction; and it is the object of this paper to develop a still more general theory.

In passing beyond Euclidean geometry, gravitation makes its appearance; in passing beyond Riemannian geometry, electromagnetic force appears; what remains to be gained by further generalisation? Clearly, the non-Maxwellian binding-forces which hold together an electron. But the problem of the electron must be difficult, and I cannot say whether the

present generalisation succeeds in providing the materials for its solution. The present paper does not seek these unknown laws, but aims at consolidating the known laws. I hope to show that, in freeing Weyl's geometry from its limitation, the whole scheme becomes simplified, and new light is thrown on the origin of the fundamental laws of physics.

The essential point of Weyl's theory† is that comparisons of length at different times and places may yield discordant results according to the route of comparison. Accordingly, a particular standard of length should only be used at the time and place where it is; and it is necessary to set up a separate unit of length (interval-length) at every point of space and time. Such a system of unit-standards is called a gauge-system, and Weyl's theory proceeds on the view that the gauge-system is arbitrary, just as the coordinate system is. Nevertheless, he recognises that a certain natural gauge-system plays a very important part in physics. We *do* compare lengths on the sun and the earth: and, although our method of comparison may be ambiguous for the thirtieth significant figure, we have no hesitation in deciding whether a given length on the sun should be called an inch or a mile. This natural gauge-system seems at first like an excrescence on the purely geometrical theory, diverting it from its natural development, but necessary in order to bring it into touch with physics. The theory here developed seems to throw new light on the meaning of this natural gauge-system.

A deductive study of world-geometry and an inductive study of observational science attack the problem of Nature from opposite ends. The most elementary concepts of experimental knowledge may not appear until a late stage in the deductive treatment. Our work thus falls into two stages: first, the development of a pure geometry of a very general kind; second, a physical theory based on the identification of the geometrical functions with quantities obtained by experimental measurement. The introduction of the natural gauge-system marks the transition from pure geometry to physics.

Geometrical Theory.

2. Some of the tensors used in the analysis possess the property of being unchanged by any alteration of gauge-system, in addition to their ordinary properties with regard to transformation of co-ordinates. I distinguish these by the prefix "in-" (in-tensor, in-vector, in-invariant). In symbolic notation, a star (*) indicates quantities generalised from the corresponding quantities

† 'Berlin. Sitzungsberichte,' May 30, 1918. Further developments are scattered throughout his book 'Raum, Zeit, Materie.' See also his article, 'Nature,' February 17, 1921. I need scarcely say that I am under great obligations to these sources.

in Riemannian geometry so as to maintain this "in-" property, wholly or partly.

A displacement, dx_μ, is a simple example of an in-vector. Its components are differences of coordinates (pure numbers), and have no reference to any gauge. Since dx_μ is contravariant, we often write it with the notation $(dx)^\mu$, which reminds us of its character.

Let A^μ be an in-vector representing an infinitesimal displacement at a point P. Can we find at an infinitely near point, P′, an exactly equivalent displacement? Clearly, the equivalent displacement will not in general be represented by A^μ, because, for example, in a *uniform* field of force the polar components vary from point to point. There will be a spurious change of A^μ, attributable to the curvilinearity of the coordinates, even when the in-vector is displaced "without absolute change." The most general possible linear expression for this spurious change, when the in-vector is moved through a displacement, dx_ν, is

$$-\Gamma_{\nu\alpha}^{\mu} A^\alpha (dx)^\nu, \tag{2·1}$$

where $\Gamma_{\nu\alpha}^{\mu}$, which is not assumed to be a tensor, represents arbitrary coefficients, and the expression is summed according to the usual convention. Only linear terms need be taken, because A^α and $(dx)^\nu$ are both infinitesimals.

Following Levi-Civita, displacement "without absolute change" is called *parallel displacement*. For parallel displacement, the change dA^μ must be equal to (2·1); and accordingly

$$\frac{\partial A^\mu}{\partial x_\nu} + \Gamma_{\nu\alpha}^{\mu} A^\alpha = 0, \tag{2·2}$$

is the condition for parallel displacement.

The definiteness of the notion of parallel displacement depends on the recognition of some kind of equivalence between displacements at infinitely near points—not between the analytical expressions, but between the relations which they express. We assume that there is such an equivalence; and, in justification, we point out that it seems to be the minimum assumption possible. For if there were no comparability of relations, even the most closely adjacent, the continuum would be divested of even the rudiments of structure, and nothing in nature could resemble anything else.

If in (2·1) we take A^α to be a displacement $PP_1 = (\delta x)^\alpha$, which on parallel displacement to P′ becomes $P'P_1'$, the difference of co-ordinates of P_1' and P will be

$$(dx)^\mu + (\delta x)^\mu - \Gamma_{\nu\alpha}^{\mu} (\delta x)^\alpha (dx)^\nu. \tag{2·3}$$

Interchanging the two displacements, *i.e.*, displacing PP′ along PP_1, we shall not arrive at the same point P_1', unless

$$\Gamma_{\nu\alpha}^{\mu} = \Gamma_{\alpha\nu}^{\mu}. \tag{2·4}$$

This is a necessary condition for what is called *affine* geometry (fully treated in Weyl's 'Raum, Zeit, Materie'). It appears to express the condition that the world is "flat in its smallest parts" or that it possesses a definite tangent. I shall adopt the condition (2·4) in the present investigation, ruling out an "infinitely crinkled" world.†

3. Let an in-vector A^μ be carried by parallel displacement round a small complete circuit. The difference of the final and initial values is

$$\begin{aligned}[A^\mu]^{\text{final}}_{\text{initial}} &= \int \frac{\partial A^\mu}{\partial x_\nu}\,dx_\nu \\ &= -\int \Gamma_{\nu\alpha}^{\mu} A^\alpha (dx)^\nu \text{ by (2·2)} \\ &= -\tfrac{1}{2}\iint \left\{\frac{\partial}{\partial x_\sigma}(\Gamma_{\nu\alpha}^{\mu}A^\alpha) - \frac{\partial}{\partial x_\nu}(\Gamma_{\sigma\alpha}^{\mu}A^\alpha)\right\} dS^{\nu\sigma}, \qquad (3·1)\end{aligned}$$

by Stokes's theorem. The double integral is taken over any surface bounded by the circuit, and $dS^{\nu\sigma}$ is the antisymmetrical tensor representing the element of surface. The factor $\frac{1}{2}$ is needed because each component of surface appears twice in the summation, *e.g.*, as dS^{12} and $-dS^{21}$.

The integrand is equal to

$$\begin{aligned}&A^\alpha\left(\frac{\partial}{\partial x_\sigma}\Gamma_{\nu\alpha}^{\mu} - \frac{\partial}{\partial x_\nu}\Gamma_{\sigma\alpha}^{\mu}\right) + \Gamma_{\nu\alpha}^{\mu}\frac{\partial A^\alpha}{\partial x_\sigma} - \Gamma_{\sigma\alpha}^{\mu}\frac{\partial A^\alpha}{\partial x_\nu} \\ &\qquad = A^\alpha\left(\frac{\partial}{\partial x_\sigma}\Gamma_{\nu\alpha}^{\mu} - \frac{\partial}{\partial x_\nu}\Gamma_{\sigma\alpha}^{\mu}\right) - \Gamma_{\nu\alpha}^{\mu}\Gamma_{\sigma\rho}^{\alpha}A^\rho + \Gamma_{\sigma\alpha}^{\mu}\Gamma_{\nu\rho}^{\alpha}A^\rho \text{ by (2·2).}\end{aligned}$$

Changing the dummy suffix from α to ρ in the first term, this becomes

$$-{}^*B_{\rho\nu\sigma}^{\mu}A^\rho,$$

where
$$ {}^*B_{\rho\nu\sigma}^{\mu} = -\frac{\partial}{\partial x_\sigma}\Gamma_{\nu\rho}^{\mu} + \frac{\partial}{\partial x_\nu}\Gamma_{\sigma\rho}^{\mu} + \Gamma_{\nu\alpha}^{\mu}\Gamma_{\sigma\rho}^{\alpha} - \Gamma_{\sigma\alpha}^{\mu}\Gamma_{\nu\rho}^{\alpha}. \qquad (3·2)$$

Hence
$$[A^\mu] = \tfrac{1}{2}\iint {}^*B_{\rho\nu\sigma}^{\mu}A^\rho dS^{\nu\sigma}. \qquad (3·3)$$

In Einstein's theory, even without Weyl's generalisation, the integrand does not usually vanish, and the initial and final values of A^μ are unequal. Of course, the same discrepancy appears if A^μ is displaced by different routes

† A coordinate-system enumerates the points of a continuum in particular *order* and this order is unchanged by any transformation of coordinates. If the world referred to an arbitrary coordinate-system turns out to be infinitely crinkled, I think this must indicate a disharmony between the coordinate-order and the structural order of the points. The remedy is to change the coordinate enumeration, not by continuous transformation, but by re-sorting the points. Evidently our displacement dx_μ will not be the measure of any physical (structural) relation unless some preliminary agreement of order is postulated. The problem of determining the conditions of this preliminary agreement is essentially that solved by A. A. Robb in his 'Theory of Time and Space.' If ever the theory comes to deal with essentially discontinuous phenomena (*e.g.*, quanta) the condition (2·4) may have to be reconsidered.

to another point, and the difference is shown by (3·3) to be proportional to the area of the circuit, *ceteris paribus.* The formula (3·3) applies only to small circuits for which the square of the area can be neglected, because A^ρ is ambiguous except on the boundary of the circuit where the route of transfer has been defined; but the ambiguity of A^ρ is of the same order of magnitude as $[A^\mu]$, and its surface integral becomes negligible compared with $[A^\mu]$ when the circuit is sufficiently small. Since then the circuit is infinitesimal, we can omit the integral sign, obtaining

$$[A^\mu] = \tfrac{1}{2}{}^*B_{\rho\nu\sigma}{}^\mu A^\rho dS^{\nu\sigma}. \tag{3·4}$$

Now $[A^\mu]$ is the difference of two vectors at the same point, and is therefore a vector; $dS^{\nu\sigma}$ is a tensor; hence ${}^*B_{\rho\nu\sigma}{}^\mu$ is a tensor. Moreover, it is an in-tensor, since there has been as yet no reference to any gauge.

We obtain another in-tensor by setting $\mu = \sigma$, *i.e.*, contracting,

$${}^*G_{\rho\nu} = {}^*B_{\rho\nu\sigma}{}^\sigma = -\frac{\partial}{\partial x_\sigma}\Gamma_{\nu\rho}{}^\sigma + \frac{\partial}{\partial x_\nu}\Gamma_{\sigma\rho}{}^\sigma + \Gamma_{\nu\alpha}{}^\sigma\Gamma_{\sigma\rho}{}^\alpha - \Gamma_{\sigma\alpha}{}^\sigma\Gamma_{\nu\rho}{}^\alpha. \tag{3·5}$$

These two in-tensors express intrinsic properties of the continuum. So far as I can see no others (except products of these) exist. The first is a description as complete as possible of the structure of the continuum (the inter-connection of relations) so far as it can be studied by these methods; the second is an abbreviated summary of the information contained in the first.

We shall find later that physics has in the main contented itself with studying the abridged edition of the book of nature; only in the problems of electron structure does it seem necessary to search out the more extensive information contained in the original ${}^*B_{\rho\nu\sigma}{}^\mu$.

4. In order to introduce the $g_{\mu\nu}$ we must adopt a definite, but at present arbitrary, system of gauges. The length (interval length) l of a displacement A^μ is defined by

$$l^2 = g_{\mu\nu}A^\mu A^\nu, \tag{4·1}$$

in agreement with Einstein's fundamental equation

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu.$$

We take $g_{\mu\nu}$ to be a tensor in order to make l^2 an invariant. The tensor $g_{\mu\nu}$ may be selected quite arbitrarily,† and then equation (4·1) shows how a length, l, is to be assigned to any displacement, dx_μ, given in coordinate-differences. Different choices of $g_{\mu\nu}$ will give different plans of assigning length—in short, different gauge-systems. Length here is still a geometrical convention rather than a physical conception. So long as we have no independent means of comparing lengths, no inconsistency can possibly be

† For convenience, however, we limit the choice to tensors symmetrical in μ and ν.

detected in our scheme. When we come to identify these geometrical lengths with physical lengths, which can be compared by experimental methods, conditions must be introduced which will limit our choice of $g_{\mu\nu}$. But that will come later; the present analysis applies to any gauge-system whatever.

Move the displacement A^μ by parallel displacement through dx_σ. Differentiating (4·1) we have

$$d(l^2) = \left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma} A^\mu A^\nu + g_{\mu\nu} A^\nu \frac{\partial A^\mu}{\partial x_\sigma} + g_{\mu\nu} A^\mu \frac{\partial A^\nu}{\partial x_\sigma}\right) dx_\sigma$$

$$= \left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma} A^\mu A^\nu - g_{\mu\nu} \Gamma_{\sigma\alpha}{}^{\mu} A^\nu A^\alpha - g_{\mu\nu} \Gamma_{\sigma\alpha}{}^{\nu} A^\mu A^\alpha\right) dx_\sigma \text{ by (2·2)}$$

$$= \left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - g_{\alpha\nu} \Gamma_{\sigma\mu}{}^{\alpha} - g_{\mu\alpha} \Gamma_{\sigma\nu}{}^{\alpha}\right) A^\mu A^\nu (dx)^\sigma,$$

by interchanging dummy suffixes.

In conformity with the usual rule for lowering suffixes, we write

$$\Gamma_{\sigma\mu,\nu} = g_{\alpha\nu} \Gamma_{\sigma\mu}{}^{\alpha},$$

so that

$$d(l^2) = \left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \Gamma_{\sigma\mu,\nu} - \Gamma_{\sigma\nu,\mu}\right) A^\mu A^\nu (dx)^\sigma. \tag{4·2}$$

But $d(l^2)$, the difference of two invariants, is an invariant. Hence the quantity in the bracket is a tensor. We denote it by $2K_{\mu\nu,\sigma}$. Thus

$$2K_{\mu\nu,\sigma} = \frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \Gamma_{\sigma\mu,\nu} - \Gamma_{\sigma\nu,\mu}. \tag{4·3}$$

Similarly

$$2K_{\mu\sigma,\nu} = \frac{\partial g_{\mu\sigma}}{\partial x_\nu} - \Gamma_{\nu\mu,\sigma} - \Gamma_{\nu\sigma,\mu},$$

$$2K_{\nu\sigma,\mu} = \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \Gamma_{\mu\nu,\sigma} - \Gamma_{\mu\sigma,\nu}.$$

Adding these and subtracting (4·3), we have

$$\Gamma_{\mu\nu,\sigma} = \tfrac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) - K_{\mu\sigma,\nu} - K_{\nu\sigma,\mu} + K_{\mu\nu,\sigma}. \tag{4·4}$$

We write
$$S_{\mu\nu,\sigma} = K_{\mu\nu,\sigma} - K_{\mu\sigma,\nu} - K_{\nu\sigma,\mu}, \tag{4·5}$$

where $S_{\mu\nu,\sigma}$ is evidently a tensor symmetrical in μ and ν. Then, raising the suffixes by multiplying through by $g^{\sigma\tau}$, (4·4) becomes

$$\Gamma_{\mu\nu}{}^{\tau} = \{\mu\nu, \tau\} + S_{\mu\nu}{}^{\tau}. \tag{4·6}$$

We see that $\Gamma_{\mu\nu}{}^{\tau}$ is a generalisation of the Christoffel symbol, and might appropriately be denoted by $^*\{\mu\nu, \tau\}$. Though not a tensor, it possesses the "in-" property, since it was introduced prior to any gauge system. We note that $K_{\mu\nu,\sigma}$ is not an in-tensor, since $d(l^2)$ depends on the gauge.

It is here that we make an important divergence from Weyl's geometry; his theory is obtained by giving to $K_{\mu\nu,\sigma}$ the special form $g_{\mu\nu}\phi_\sigma$.

5. From (3·2) and (4·6) we can evaluate ${}^*B_{\mu\nu\sigma}{}^\rho$. If the terms in $S_{\mu\nu}{}^\tau$ are dropped, we obtain immediately the usual expression for the Riemann-Christoffel tensor $B_{\mu\nu\sigma}{}^\rho$ [Report,† equation (23)]. The additional terms are

$$\frac{\partial}{\partial x_\nu} S_{\mu\sigma}{}^\rho - \frac{\partial}{\partial x_\sigma} S_{\mu\nu}{}^\rho + \{\mu\sigma, \alpha\} S_{\nu\alpha}{}^\rho + \{\nu\alpha, \rho\} S_{\mu\sigma}{}^\alpha - \{\mu\nu, \alpha\} S_{\sigma\alpha}{}^\rho$$
$$- \{\sigma\alpha, \rho\} S_{\mu\nu}{}^\alpha + S_{\nu\alpha}{}^\rho S_{\mu\sigma}{}^\alpha - S_{\sigma\alpha}{}^\rho S_{\mu\nu}{}^\alpha.$$

The first six terms easily reduce to

$$(S_{\mu\sigma}{}^\rho)_\nu - (S_{\mu\nu}{}^\rho)_\sigma,$$

where the final suffix denotes covariant differentiation according to the ordinary rule, viz.,

$$(S_{\mu\sigma}{}^\rho)_\nu = \frac{\partial}{\partial x_\nu} S_{\mu\sigma}{}^\rho - \{\mu\nu, \alpha\} S_{\alpha\sigma}{}^\rho - \{\sigma\nu, \alpha\} S_{\mu\alpha}{}^\rho + \{\alpha\nu, \rho\} S_{\mu\sigma}{}^\alpha.$$

Hence

$$ {}^*B_{\mu\nu\sigma}{}^\rho = B_{\mu\nu\sigma}{}^\rho + (S_{\mu\sigma}{}^\rho)_\nu - (S_{\mu\nu}{}^\rho)_\sigma + S_{\nu\alpha}{}^\rho S_{\mu\sigma}{}^\alpha - S_{\sigma\alpha}{}^\rho S_{\mu\nu}{}^\alpha. \quad (5·1)$$

Setting $\rho = \sigma$, we have

$$ {}^*G_{\mu\nu} = G_{\mu\nu} + 2\kappa_{\mu\nu} - (S_{\mu\nu}{}^\sigma)_\sigma + S_{\alpha\nu}{}^\beta S_{\beta\mu}{}^\alpha - 2\kappa_\alpha S_{\mu\nu}{}^\alpha, \quad (5·2)$$

where we have written

$$2\kappa_\mu = S_{\sigma\mu}{}^\sigma, \quad (5·3)$$

and $\kappa_{\mu\nu}$ is the covariant derivative of κ_μ. Again, multiplying by $g^{\mu\nu}$

$$ {}^*G = G + 2\kappa^\nu{}_\nu + \lambda^\nu{}_\nu + S_{\nu\alpha,\beta} S^{\nu\beta,\alpha} + 2\kappa_\alpha\lambda^\alpha \quad (5·4)$$

where

$$\lambda_\mu = -S^\sigma{}_{\sigma,\mu}. \quad (5·5)$$

(The distinction between (5·3) and (5·5) is that in $S_{\sigma\mu}{}^\sigma$ μ occupies one of the two symmetrical places, and in $S^\sigma{}_{\sigma,\mu}$ it occupies the third unsymmetrical place. κ_μ and λ_μ are quite different vectors.)

The only term on the right of (5·2) which is not symmetrical in μ and ν is $2\kappa_{\mu\nu}$, which may be written $(\kappa_{\mu\nu} + \kappa_{\nu\mu}) + (\kappa_{\mu\nu} - \kappa_{\nu\mu})$. Thus, if we write

$$ {}^*G_{\mu\nu} = R_{\mu\nu} + F_{\mu\nu}, \quad (5·6)$$

where $R_{\mu\nu}$ is symmetrical and $F_{\mu\nu}$ antisymmetrical, we have

$$\begin{aligned} F_{\mu\nu} &= \kappa_{\mu\nu} - \kappa_{\nu\mu} \\ &= \frac{\partial\kappa_\mu}{\partial x_\nu} - \frac{\partial\kappa_\nu}{\partial x_\mu}, \end{aligned} \quad (5·7)$$

the covariant curl being the same as the ordinary curl for a *covariant* vector.

† 'Report on the Relativity Theory of Gravitation,' Physical Society (Fleetway Press).

Since $R_{\mu\nu}$ is half the sum of $^*G_{\mu\nu}$ and $^*G_{\nu\mu}$, and $F_{\mu\nu}$ is half the difference, both $R_{\mu\nu}$ and $F_{\mu\nu}$ are in-tensors.†

Similarly we can divide $^*B_{\mu\nu\sigma\rho}$ into two parts

$$^*B_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} + F_{\mu\nu\sigma\rho}, \tag{5·8}$$

where R is antisymmetrical and F symmetrical in μ and ρ. (The whole tensor is antisymmetrical in ν and σ.) It is easy to show that

$$F_{\mu\nu\sigma\rho} = (K_{\mu\rho,\nu})_\sigma - (K_{\mu\rho,\sigma})_\nu. \tag{5·9}$$

These, however, are not in-tensors, since the $g_{\mu\nu}$ must be introduced in order to lower the suffix ρ.

Physical Theory.

6. We now introduce the natural gauge of the world, which is determined by measures of space and time made with material and optical appliances. Any apparatus used to measure the world is itself part of the world, so that a natural gauge represents the world as self-gauging. *This can only mean that the tensor $g_{\mu\nu}$ introduced in* (4·1) *is not extraneous and arbitrary, but is a tensor already contained in the world-geometry.* There is only one such tensor available, viz., $^*G_{\mu\nu}$. Hence natural length is given by

$$l^2 = {}^*G_{\mu\nu} A^\mu A^\nu. \tag{6·1}$$

Clearly the antisymmetrical part drops out in this equation, so that

$$l^2 = R_{\mu\nu} A^\mu A^\nu.$$

Accordingly we must take

$$\lambda g_{\mu\nu} = R_{\mu\nu}, \tag{6·2}$$

introducing a universal constant, λ, for convenience, in order to remain free to use the centimetre instead of the natural unit of length.

The difference between $R_{\mu\nu}$ and $G_{\mu\nu}$ in (5·2) consists of terms formed out of the tensor $K_{\mu\nu}{}^\sigma$. We shall find later that this tensor represents electromagnetic and electronic effects, so that the more "empty" the space the more nearly does $R_{\mu\nu}$ approximate to $G_{\mu\nu}$. Hence, in an empty region, the gauging-equation (6·2) becomes

$$G_{\mu\nu} = \lambda g_{\mu\nu}. \tag{6·4}$$

This is De Sitter's equation for a spherical (or, rather, a quasi-spherical) world. The gauging-equation is, in fact, an *alias* of the law of gravitation. If λ approaches zero so that the dimensions of the world become infinitely great compared with our unit of measure, we have Einstein's original

† It may be noticed that $F_{\nu\sigma} = \frac{1}{2}{}^*B_{\rho\nu\sigma}{}^\rho$, a result easily obtained from (5·1), remembering that $B_{\rho\nu\sigma}{}^\rho = 0$. It follows fairly easily that the proportionate change of volume of a four-dimensional element carried by parallel displacement round a small circuit is $F_{\nu\sigma}\, dS^{\nu\sigma}$.

gravitational equations $G_{\mu\nu} = 0$. In later work he has used the more general form (6·4), giving a naturally curved world ('Report,' § 50).†

The form of the world given by these equations is only quasi-spherical, because it requires twenty conditions to determine a truly spherical world, just as it requires twenty conditions ($B_{\mu\nu\sigma\rho} = 0$) to determine a truly flat world. But a world conditioned by (6·4) will have the more important spherical properties. It has this symmetrical form, not from any innate tendency, but because our natural gauge of length at different points and in different directions is adapted to make it so. If we have two electrons at different places, we expect them to be of the same size; we also expect an electron to be spherical. But the equilibrium shape and size of an electron must be determined by something in the world-conditions where it is (apart possibly from small accidental variations persisting from its previous history). The electron must measure itself against something in order to find out how large it ought to be; and in empty space there seems to be nothing but the radii of curvature of the world for it to measure itself against. Similarly, any material object of specified constitution must have determined size and shape *in relation to the radii of the world.* And in so far as the measure of length rests on such bodies, the form of the world expressed in this measure is bound to be quasi-spherical.

It is only in empty space that the approximate equation (6·4) is valid. Electrons are recognised to be small regions of intense local curvature; this is provided for in the accurate gauging equation (6·2), since the difference between $R_{\mu\nu}$ and $G_{\mu\nu}$ may in certain circumstances become large. Intense curvature (or mass) must always be associated with large values of $S_{\mu\nu}{}^{\sigma}$, and we shall find later that large electric fields are associated with large values of this tensor.

Natural space and time must obey Riemannian geometry. It would be easy to devise fanciful geometries in which, for example, ds^4 was a quartic function of the coordinate-differences. But there is only one in-invariant of a simple character‡ associated with a displacement A^{μ}, viz., $*G_{\mu\nu}A^{\mu}A^{\nu}$, so that (except for far-fetched combinations) no quartic function of the coordinate-differences could have the necessary property. Even $*B_{\mu\nu\sigma}{}^{\rho}$ is no use, for we cannot counteract the upper ρ.

7. Having thus identified physical space-time, we proceed to identify "things." There are three sorts of physical attributes of things to be identified; and our plan is first to inquire what experimental properties

† The difference between Einstein's and De Sitter's theories depends on the total amount of matter in the world. Both adopt (6·4) as the condition for empty space.

‡ The next simplest in-invariant is $*B_{\mu\nu\sigma}{}^{\rho}\,*B_{\lambda\tau\rho}{}^{\sigma}\,A^{\mu}\,A^{\nu}\,A^{\lambda}\,A^{\tau}$.

they possess, and then to seek a geometrical tensor possessing these properties *by virtue of mathematical identities.* The tensors need not now be in-tensors, because our description of the experimental properties presupposes the use of the natural gauge-system; and there is no reason to think it would hold good for other gauge-systems:—

(1) The energy-tensor T_μ^ν comprises the energy, momentum, and stress in a region. These must satisfy the condition of conservation of energy and momentum, which is expressed by

$$T_{\mu}{}^{\nu}{}_{\nu} = 0. \tag{7·1}$$

This enables us to make the identification ('Report,' § 39)

$$-8\pi T_\mu^\nu = G_\mu^\nu - \tfrac{1}{2} g_\mu^\nu (G - 2\lambda), \tag{7·2}$$

which satisfies (7·1) identically. Here λ might be any constant so far as conservation is concerned. We consider, however, that the energy-tensor should be something which vanishes altogether in empty space (containing no electromagnetic fields). If $T_\mu^\nu = 0$, we have, by contracting (7·2),

$$G = 4\lambda$$

so that

$$0 = G_\mu^\nu - g_\mu^\nu \lambda,$$

or lowering the suffix

$$G_{\mu\nu} = \lambda g_{\mu\nu},$$

agreeing with (6·4) if λ is the same constant. Thus, from the conservation of energy and momentum, we deduce the law of gravitation once more.

(2) The electromagnetic force-tensor $F_{\mu\nu}$ must satisfy two of Maxwell's equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F_{\sigma\mu}}{\partial x_\nu} = 0. \tag{7·3}$$

This will be an identity if $F_{\mu\nu}$ is the curl of a vector. This leads naturally to an identification with the in-tensor already called $F_{\mu\nu}$, which we have seen is the curl of a vector κ_μ. Hence κ_μ will be the electromagnetic potential.

(3) The electric charge-and-current vector J^μ must satisfy the condition of conservation of electric charge, viz., its divergence $J^\mu{}_\mu$ vanishes. This will be an identity if J^μ is itself the divergence of any antisymmetrical contravariant tensor. Accordingly, we identify

$$J^\mu = F^{\mu\nu}{}_\nu. \tag{7·4}$$

Additional confirmation is given by the fact that (7·4) represents Maxwell's two remaining equations ('Report,' § 45).

It would in any case be reasonable to make these identifications—the simplest possible—checking them by comparison with observation later on. But actually very little choice is possible; that is to say, it is not easy to

find any other geometrical tensors having the necessary properties as identities. Further, we can scarcely expect much further experimental test, since (with one exception†) all the known laws of mechanics and electromagnetism are already satisfied.

As regards alternatives, some other tensor functions of $g_{\mu\nu}$ are known which satisfy (7·1) identically: but these do not agree with the condition for empty space, $G_{\mu\nu} = \lambda g_{\mu\nu}$. They are extremely complicated expressions, involving derivatives of the fourth or higher orders. An alternative identification of the electromagnetic force with the curl of λ_μ (instead of the curl of κ_μ) seems possible, though rather perverse—seeing that it would leave the fundamental in-tensor $F_{\mu\nu}$ apparently doing nothing to justify its existence.

8. The gauging-equation (6·2) can also be demonstrated from optical theory. The light pulse diverging from a point occupies a conical locus in four dimensions, and must therefore satisfy an equation of the form $a_{\mu\nu} dx_\mu dx_\nu = 0$. This unique locus exists independently of gauge or coordinates, and therefore there must be an in-invariant equation defining it. Clearly $a_{\mu\nu}$ must be an in-tensor, which leaves as the only possibility $a_{\mu\nu} = {}^*G_{\mu\nu}$, or

$$R_{\mu\nu} dx_\mu dx_\nu = 0. \tag{8·1}$$

But in Einstein's theory the light-cone is

$$g_{\mu\nu} dx_\mu dx_\nu = 0, \tag{8·2}$$

so that Einstein's gauge is given by

$$R_{\mu\nu} = \lambda g_{\mu\nu}.$$

Our gauging-equation is therefore certainly true wherever light is propagated, *i.e.*, everywhere except inside the electron. Who shall say what is the ordinary gauge inside the electron? It should, however, be noticed that this optical treatment does not prove that λ is a universal constant. If it were a varying function of position, (8·1) and (8·2) would still agree. The constancy of λ depends on the arguments previously given.

If a displacement A^μ is taken by parallel displacement round a small complete circuit, its change of length is

$$\begin{aligned} \delta l^2 &= \delta (g_{\mu\rho} A^\mu A^\rho) = \delta (A_\mu A^\mu) = 2 A_\mu \delta A^\mu \\ &= A_\mu {}^*B_{\rho\nu\sigma}{}^\mu A^\rho dS^{\nu\sigma} \text{ by } (3·4) \\ &= (R_{\rho\nu\sigma\mu} + F_{\rho\nu\sigma\mu}) A^\mu A^\rho dS^{\nu\sigma} \text{ by } (5·8) \\ &= F_{\rho\nu\sigma\mu} A^\mu A^\rho dS^{\nu\sigma}, \end{aligned} \tag{8·3}$$

since $R_{\rho\nu\sigma\mu}$ is antisymmetrical in μ and ρ.

† The law of ponderomotive force of an electromagnetic field, see § 9.

In Weyl's theory $F_{\rho\nu\sigma\mu}$ is of the form $g_{\rho\mu}F_{\nu\sigma}$ so that (8·3) reduces to

$$\delta l^2 = F_{\nu\sigma}dS^{\nu\sigma} \,.\, l^2, \tag{8·4}$$

so that, although a vector in general alters length on describing a circuit, zero-length is unaltered on Weyl's theory. Accordingly, for him zero-length is unique and involves no specification of a route of comparison; whereas in my theory zero-length may not remain zero on parallel displacement according to (8·3). This was for a long while my chief obstacle in generalising his theory, since the abandonment of the uniqueness of zero-length might well seem to leave no unique track for the propagation of light. But we see that the difficulty is surmounted; the unique light-cone is given by the only simple in-invariant equation that can be formed; and the condition that it corresponds to zero-length is one of the factors which determine our natural gauge-system.

9. The ponderomotive force of an electromagnetic field on matter containing electric charges is

$$k_\mu = F_{\mu\nu}J^\nu, \tag{9·1}$$

('Report,' equation (46·1)). This is the tensor-generalisation of the elementary law

$$m\frac{d^2x}{dt^2} = eX, \tag{9·2}$$

which follows immediately from the usual definition of electric force. We have to show that (9·1) can also be deduced when the electric force is defined by the method of identification in § 7.

It will be sufficient to consider an isolated electron. If we consider the experimental verification of (9·1) or (9·2) by the deflection of an electron travelling in an electromagnetic field, it is clear that the field $F_{\mu\nu}$ referred to in (9·1) is the applied external field, and no account is taken of any disturbance of this field caused by the accelerated electron itself. We denote this external field by $F'_{\mu\nu}$. Also, for a particle,

$$\begin{aligned} k^\mu &= -m\left\{\frac{d^2x_\mu}{ds^2} + \{\alpha\beta,\mu\}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}\right\} \\ &= -mA^\nu A^\mu{}_\nu, \end{aligned} \tag{9·3}$$

where $A^\mu = dx_\mu/ds$ ('Report,' § 29). If in (9·3) we put $k_\mu = 0$, we obtain the usual equations for the track of an undisturbed particle, *i.e.*, a geodesic, which can be deduced directly from the conservation of energy and momentum.

The equation to be proved is thus

$$mA^\nu A_{\mu\nu} = -F_{\mu\nu}'J^\nu. \tag{9·4}$$

In the ordinary text-books it is proved that the potentials due to a slowly-moving elementary charge de are

$$F, G, H, \Phi = \frac{de}{r}\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right).$$

These expressions are deduced solely from Maxwell's equations (7·3) and (7·4), and we need not here trouble about retarded values, since our application is to the case when r is very small. The general vector-equations are evidently

$$\kappa^{\mu} = \frac{de}{r} A^{\mu}, \tag{9·5}$$

where de/r is treated as an invariant, that is to say, its value is to be calculated for Galilean axes with respect to which the electron is momentarily at rest. Hence the electric force *due to the electron* is

$$F_{\mu\nu} = \kappa_{\mu\nu} - \kappa_{\nu\mu} = (A_{\mu\nu} - A_{\nu\mu}) \int \frac{de}{r},$$

since all parts of the electron have the same velocity A^{μ}.

Now, suppose that the electron moves in such a way that its own field on the average just neutralises the external field $F_{\mu\nu}'$ in the region occupied by the electron. The average value of $F_{\mu\nu}$ for all the elements of charge constituting the electron will be given by

$$\begin{aligned} e \times \text{average } F_{\mu\nu} &= (A_{\mu\nu} - A_{\nu\mu}) \iint \frac{de_1 de_2}{r_{12}} \\ &= (A_{\mu\nu} - A_{\nu\mu})\, e^2/a, \end{aligned}$$

where $1/a$ is an average value of $1/r$ taken for every pair of points in the electron. Also $J^{\mu} = eA^{\mu}$. Hence, multiplying by A^{ν}, we shall have, if the external field is neutralised,

$$F_{\mu\nu}' J^{\nu} = -A^{\nu}(A_{\mu\nu} - A_{\nu\mu})\, e^2/a.$$

But $$A^{\nu} A_{\nu\mu} = A^{\nu}{}_{\mu} A_{\nu} = \tfrac{1}{2}(A^{\nu} A_{\nu})_{\mu} = 0,$$

since $$A^{\nu} A_{\nu} = g_{\mu\nu} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} = 1.$$

Hence $$F_{\mu\nu}' J^{\nu} = -A^{\nu} A_{\mu\nu} e^2/a, \tag{9·6}$$

which agrees with (9·4) if the mass of the electron is

$$m = e^2/a. \tag{9·7}$$

It is now evident that (9·1) implies that an electron is a structure which cannot exist in a resultant field of electromagnetic force. Further than this we cannot expect to go without a detailed analysis of electron structure. It is a question of the adjustment of the electron to the world-conditions in

which it finds itself. The electron evidently has a certain amount of adaptability to different world-conditions; and can always resist differential electromagnetic force, just as a static electron can resist the radial electrostatic force due to its own charge. Knowing nothing of the laws of electron structure, the existence of an isolated electron at rest in no external field is apparently a "miracle"; our calculation shows that the existence of an electron having the acceleration given by (9·1) in an external electromagnetic field is *precisely the same miracle* viewed from another standpoint.

There is a close resemblance between this method of deducing the acceleration of an electron in an electric field and that of a material body in a gravitational field by the principle of equivalence. Neither body can be in equilibrium (*i.e.*, exist) if there is any resultant field of force relative to it; it would be out of adjustment with the surrounding state of the world, and the hypothesis represents a self-contradictory structure. Therefore the state of motion must be such that no resultant field of force exists where the object is; the body and the electron destroy the field by their own accelerations, and this criterion gives the motion in each case. Both, however, are able in some way to oppose differential forces by internal stresses.

The electric mass m is not immediately a source of gravitational attraction, so far as the foregoing discussion can show. But an electron *has* gravitational mass; it is by hypothesis a place where $F_{\mu\nu}$, and therefore $S_{\mu\nu}^{\sigma}$, has very large components. Reference to (5·4) shows that in this case G will differ very considerably from R (unless there is an unsuspected cancelling of terms), showing that the electron is a region of abnormal world-curvature, and therefore the source of gravitational effects. From the constancy of structure of electrons, the gravitational mass of a pair of electrons (positive and negative) will be in a constant ratio to their electric mass. We are under no obligation to prove that the ratio is the same for the positive and negative electrons separately; because there is no observational evidence for this. In § 11, however, it appears likely that the ratio of the electrical to the gravitational mass is $8\pi\lambda$; if so, it may be presumed that the factor applies to positive and negative electrons independently.

From (9·1) the expression for the electromagnetic energy-tensor is deduced as explained in 'Report,' § 46.

Further Developments.

10. The whole energy-tensor is given by

$$-8\pi T_{\mu}^{\nu} = \lambda \left(G_{\mu}^{\nu} - \tfrac{1}{2} g_{\mu}^{\nu} (G - 2\lambda)\right), \tag{10·1}$$

and the electromagnetic energy-tensor by

$$E_{\mu}^{\nu} = -F_{\mu\sigma} F^{\nu\sigma} + \tfrac{1}{4} g_{\mu}^{\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{10·2}$$

('Report,' equations (50·4) and (46·4)). It is necessary to insert the factor λ in (10·1) in order to make the dimensions consistent. Even now it is not certain that the two expressions are in common units; but, at any rate, their ratio will not be altered by changing the unit of length, *i.e.*, changing λ.

Contracting (10·2), $E = 0$ identically. It is thus impossible to build up matter or electrons from electromagnetic fields alone, and some other form of energy must be present. This represents the non-Maxwellian binding forces. It is convenient to write

$$R_{\mu\nu} = G_{\mu\nu} + H_{\mu\nu}, \qquad R = G + H, \tag{10·3}$$

so that from (5·2) and (5·4)

$$H_{\mu\nu} = \kappa_{\mu\nu} + \kappa_{\nu\mu} - (S_{\mu\nu}{}^{\sigma})_{\sigma} + \{S_{\alpha\nu}{}^{\beta} S_{\beta\mu}{}^{\alpha} - 2\kappa_{\alpha} S_{\mu\nu}{}^{\alpha}\}, \tag{10·4}$$

$$H = 2\kappa^{\alpha}{}_{\alpha} + \lambda^{\alpha}{}_{\alpha} + \{S_{\alpha\beta,\gamma} S^{\alpha\gamma,\beta} + 2\kappa_{\alpha}\lambda^{\alpha}\}. \tag{10·5}$$

Contracting (10·1)

$$-8\pi T = \lambda(-G + 4\lambda).$$

But

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \,.\, \lambda g_{\mu\nu} = 4\lambda.$$

Hence

$$-8\pi T = \lambda(R - G) = \lambda H, \tag{10·6}$$

and similarly

$$8\pi T_{\mu}{}^{\nu} = \lambda(H_{\mu}{}^{\nu} - \tfrac{1}{2} g_{\mu}{}^{\nu} H). \tag{10·7}$$

We may regard these formulæ as giving the electrical aspect of material energy relating it to the tensor $S_{\mu\nu}{}^{\sigma}$, as opposed to (10·1), which shows its gravitational aspect.

To fix ideas as to the order of magnitude of terms, let us take the centimetre as unit of length, and choose approximately Galilean coordinates with a centimetre mesh. In these units λ (the curvature of the world in empty space) can scarcely be greater than 10^{-50}. Calling this a small quantity of the first order, it is easily seen that $S_{\mu\nu}{}^{\sigma}$ is of the first order, except in electric fields so intense as to alter the order of magnitude of the local curvature. Accordingly, the expressions for $H_{\mu\nu}$ and H break up into first-order and second-order terms, the latter being bracketed in (10·4) and (10·5). Evidently the formation of electrons of definite size is associated with the non-linearity of these equations, the product terms becoming important when the electric field is sufficiently intense.

For an electron at rest $\kappa^{\alpha}{}_{\alpha}$ vanishes. Although the physical interpretation of λ^{α} is not known, it seems likely that $\lambda^{\alpha}{}_{\alpha}$ will also vanish in the static case. Thus the mass of an isolated electron is probably represented entirely by the product terms in H.

11. It is natural to infer from (10·1) and (10·2) that in space containing no electrons $T_{\mu}{}^{\nu}$ and $E_{\mu}{}^{\nu}$, if reduced to a common unit, must be equal. But this involves an arbitrary localisation of the energy-tensors which is scarcely

justified, since the connection established between them *viâ* (9·1) refers to the integrated expressions.

The difficulty of localisation can be realised by considering an electron at rest. In this case its relative mass is equal to its invariant mass. On the ordinary view the relative mass is attributed to the inertia of the electromagnetic field which surrounds it; but the field contributes nothing to the invariant mass. The two masses are equal, if not synonymous, yet one is arbitrarily located in the field and the other in the electron.

From the formulæ already given it is possible to express T_μ^ν and E_μ^ν in terms of the S or K tensor; but no obvious connection between them can be found. It is possible that the formulæ may admit of some simplification, since the gauging equation must introduce some relations between the components of $K_{\mu\nu}^\sigma$, which, however, are difficult to follow out. The following discussion seems to arrive at a natural connection between T_μ^ν and E_μ^ν, and it appears to have a fundamental significance.

Consider the invariant

$$U = \int {}^*G_{\mu\nu}{}^*G^{\nu\mu}\sqrt{(-g)}\,d\tau, \tag{11·1}$$

which, though not strictly an in-invariant, is independent of the value of λ, and is accordingly a pure number.† We have

$$\begin{aligned} {}^*G_{\mu\nu}{}^*G^{\nu\mu} &= (R_{\mu\nu}+F_{\mu\nu})(R^{\nu\mu}+F^{\nu\mu}) \\ &= R_{\mu\nu}R^{\mu\nu}-F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{11·2}$$

owing to the symmetry of $R_{\mu\nu}$ and antisymmetry of $F_{\mu\nu}$.

Calculate the variation of U due to small arbitrary variations of the $g_{\mu\nu}$ which vanish at and near the boundary of the region considered. (These variations represent alterations of the state of the world expressed by the in-tensor $R_{\mu\nu}$, which lead to corresponding variations of the $g_{\mu\nu}$ through the gauging-equation.) We have

$$\begin{aligned} \delta[R_{\mu\nu}R^{\mu\nu}\sqrt{(-g)}] &= R_{\mu\nu}\delta[R^{\mu\nu}\sqrt{(-g)}] \\ &\qquad + R^{\mu\nu}\delta[R_{\mu\nu}\sqrt{(-g)}] - R_{\mu\nu}R^{\mu\nu}\delta[\sqrt{(-g)}] \\ &= \lambda\{g_{\mu\nu}\delta[R^{\mu\nu}\sqrt{(-g)}]+g^{\mu\nu}\delta[R_{\mu\nu}\sqrt{(-g)}]\} \\ &\qquad -4\lambda^2\delta[\sqrt{(-g)}], \end{aligned}$$

but $\lambda\sqrt{(-g)}(R^{\mu\nu}\delta g_{\mu\nu}+R_{\mu\nu}\delta g^{\mu\nu}) = \lambda^2\sqrt{(-g)}(g^{\mu\nu}\delta g_{\mu\nu}+g_{\mu\nu}\delta g^{\mu\nu}) = 0,$

so that, by adding,

$$\begin{aligned} \delta[R_{\mu\nu}R^{\mu\nu}\sqrt{(-g)}] &= 2\lambda\delta[R\sqrt{(-g)}]-4\lambda^2\delta[\sqrt{(-g)}] \\ &= 2\lambda\delta\{(G-2\lambda)\sqrt{(-g)}\}+2\lambda\delta[H\sqrt{(-g)}] \text{ by (10·3).} \end{aligned}$$

† As Weyl has pointed out, the existence of several numerical invariants of a simple character is a distinctive property of a *four*-dimensional continuum. The invariant U is from many points of view the most fundamental of these.

But $(G-2\lambda)\sqrt{(-g)}$ is well known as the "action-density" of the gravitational field, and its variation is reducible to†

$$\sqrt{(-g)}\,\{G_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(G-2\lambda)\}\,\delta g^{\mu\nu} = -8\pi T_{\mu\nu}\sqrt{(-g)}\,\delta g^{\mu\nu}/\lambda,$$

rejecting a part which yields a surface-integral over the boundary, since all the variations vanish there. Again,

$$\delta[F_{\mu\nu}F^{\mu\nu}\sqrt{(-g)}] = F_{\mu\nu}F_{\alpha\beta}\,\delta[g^{\alpha\mu}g^{\beta\nu}\sqrt{(-g)}]$$

since the in-tensor $F_{\mu\nu}$ is not affected by the variations here considered. It is readily found‡ that this reduces to $-2E_{\mu\nu}\sqrt{(-g)}\,\delta g^{\mu\nu}$.

Further by (10·5)

$$\delta\int H\sqrt{(-g)}\,d\tau = \delta\int\{S_{\alpha\beta,\gamma}S^{\alpha\gamma,\beta}+2\kappa_\alpha\lambda^\alpha\}\sqrt{(-g)}\,d\tau.$$

The first-order terms are omitted, since

$$(2\kappa^\alpha{}_\alpha+\lambda^\alpha{}_\alpha)\sqrt{(-g)} = \frac{\partial}{\partial x_\alpha}\{2\kappa^\alpha\sqrt{(-g)}+\lambda^\alpha\sqrt{(-g)}\},$$

by a well-known formula for the divergence of a contravariant vector. This on integration gives a surface-integral over the boundary where all variations vanish.

Collecting these results we have

$$-\tfrac{1}{2}\delta U = \int(8\pi T_{\mu\nu}-E_{\mu\nu})\,\delta g^{\mu\nu}\,.\sqrt{(-g)}d\tau-\lambda\,\delta\int\{S_{\alpha\beta,\gamma}S^{\alpha\gamma,\beta}+2\kappa_\alpha\lambda^\alpha\}\sqrt{(-g)}\,.\,d\tau. \qquad (11·3)$$

Thus, by considering the variation of what is probably the most fundamental numerical invariant, we have brought together in one formula the whole energy-tensor and the electro-magnetic energy-tensor—a result which would be difficult to accomplish in any other way. There can be little doubt that in the combination

$$8\pi T_{\mu\nu}-E_{\mu\nu}, \qquad (11·4)$$

they are expressed in a common unit, so that the difference must represent the electronic or non-Maxwellian part of the whole energy-tensor. The remaining term on the right is also concerned with electronic terms, since outside the electron the integrand is of the second order of small quantities. Hence δU must be a variation of "action" associated with electrons. It is not clear whether we are here approaching a law of Nature or whether our analysis is a device for bringing out hidden identities in the mathematical expressions; but I incline to the latter view.

† 'Raum, Zeit, Materie,' 3rd edition, pp. 205, 253.
‡ *Ibid.*, p. 197.

It may be tempting to suggest a law of stationary action that $\delta U = 0$ This, however, is impossible. If we do not split up R into G+H, we find

$$-\tfrac{1}{2}\delta U = \int (\lambda^2 g_{\mu\nu} - E_{\mu\nu})\,\delta g^{\mu\nu} \,.\, \sqrt{(-g)}\,d\tau,$$

and our proposed law of action would be inconsistent with the identity $E = 0$.

It need not surprise us that the combination (11·4) is only met with when the reduction proceeds in a particular way. As I have already indicated, the real connection is between the integrated energies, and the localisation is probably not such as to give the same connection in each element of volume. I think that the casting away of the surface-integral from $\delta[G\sqrt{(-g)}]$ has redistributed the location so as to display the agreement.

The most important result is the reduction of $T_{\mu\nu}$ and $E_{\mu\nu}$ to a common unit, implied in (11·4). $T_{\mu\nu}$ for given masses or electric fields is known in ordinary units, whereas $E_{\mu\nu}$ involves the natural units which determine the in-tensor $F_{\mu\nu}$. Thus, taking $\lambda = 10^{-50}$ as an upper limit, I find that the quantum is 10^{-114} natural units of action. A faint hope that the natural unit of action involved in this theory might prove to be identical with the quantum is thus disappointed. It appears also that $R_{\mu\nu}$ and $F_{\mu\nu}$ may be of roughly equal magnitude in electric fields of quite ordinary strength produced in the laboratory.

12. It is a natural impression that in the light of the more general theory Einstein's work must be regarded as only a close approximation. The impression is wrong, for the present paper leads to the conclusion that Einstein's postulates and deductions are exact. The natural geometry of the world (in the sense that I have defined in 'Space, Time, and Gravitation') is the geometry of Riemann and Einstein, not Weyl's generalised geometry or mine. What *we* have sought is not the geometry of actual space and time, but the geometry of the world-structure, which is the common basis of space and time and things. I have tried to emphasise this distinction by dividing the paper into Geometrical and Physical theory. The first part is the so-called "geometry" of a general kind of relation complex; for I think that any conception of *structure* (as opposed to substance) must be analysable into a complex of relations and relata, the relata having no structural significance except as the meeting point of several relations, and the relations having no significance except as connecting and ordering the relata. This part of the theory is an essay in assigning measure to such a structure. Our method is based on an axiom of parallel displacement, *i.e.*, comparability of proximate relations; this seems to be the minimum degree of comparability which permits of any differentiation of structure. We succeed in arriving at a quantitative measure of this structure, viz., $^*G_{\mu\nu}$,

which breaks up into symmetrical and antisymmetrical parts, $g_{\mu\nu}$ and $F_{\mu\nu}$, the latter being necessarily the curl of a vector. It is at this bifurcation of the theory that Einstein begins; he has recognised these two tensors from their physical manifestations, and he expresses no view as to what lies behind them. From $g_{\mu\nu}$ he builds up interval-length, and hence actual space, time, geometry, gravitation, and mechanics; from $F_{\mu\nu}$ he builds the Maxwellian theory of electricity and magnetism. His *interval* is absolute, for it is our in-invariant $R_{\mu\nu}dx_\mu dx_\nu$; so that *his work stands unaffected by the ambiguity of length-comparisons in the initial stages of our theory.* The material he employs is furnished by our theory in exact accordance with his specification.

We have further shown that there is a more comprehensive measure of the structure given by $^*B_{\mu\nu\sigma}{}^\rho$, of which $^*G_{\mu\nu}$ is an abbreviated summary. It seems obvious that even this extended tensor is by no means an exhaustive description of all that there is in the world; but presumably it (or something analogous) marks the limit of physical science, though not necessarily of the scientific method. The curious point is that, for present-day physics, the summary $^*G_{\mu\nu}$ suffices, and either the extended $^*B_{\mu\nu\sigma}{}^\rho$ has no message for us or we have failed to interpret the message. Even in § 11 the invariant U belongs to the summary description. It remains to be seen whether there is anything of observational importance omitted in the summary—anything which plays a part in the formation of negative and positive electrons, and possibly even of the quantum.